Proofs as executions

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TCS 2012 – Amsterdam

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Plan

Logic vs computation

- The *formulae as types* approach:

$$\begin{aligned} \text{formula} &\leftrightarrow \text{type} \\ \text{proof rules} &\leftrightarrow \text{primitive instructions} \\ \text{proof} &\leftrightarrow \text{program} \\ \text{normalization} &\leftrightarrow \text{evaluation} \end{aligned}$$

- The *proof search* approach:

$$\begin{aligned} \text{formula} &\leftrightarrow \text{program} \\ \text{proof rules} &\leftrightarrow \text{operational semantics} \\ \text{proof} &\leftrightarrow \text{successful run} \end{aligned}$$

A few observations

Proof normalization, aka *cut elimination*:

- the meaning of a proof is in its normal form,
- normalization is an *explicitation* procedure,
- it really wants to be confluent.

Interpretation of concurrent processes:

- the meaning is the *interaction*, the final (irreducible) state is less relevant,
- a given process may behave very differently depending on scheduling decisions.

Proofs as schedules

The principles of our interpretation:

$$\begin{aligned} \text{formula} &\leftrightarrow \text{type of interaction} \\ \text{proof rules} &\leftrightarrow \text{primitives for building schedules} \\ \text{proof} &\leftrightarrow \text{schedule for a program} \\ \text{normalization} &\leftrightarrow \text{evaluation} \end{aligned}$$

What this is not:

- *Curry-Howard*:
  proofs are not programs, but behaviours of programs
- *Proof search*:
  the dynamics is not in proof construction but in cut-elimination
- *Specification, verification*:
  only “may”-style properties can be expressed, currently

Non-determinism in concurrent processes

We consider a CCS-style process calculus.

$$
\begin{array}{rll}
P, Q := & 1 & \text{inaction} \\
 & a.P & \text{perform } a \text{ then do } P \\
 & P \mid Q & \text{interaction of } P \text{ and } Q \\
 & (\nu a)P & \text{scope restriction}
\end{array}
$$

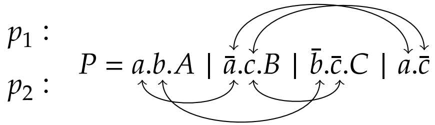
There is one source of non-determinism:
the pairing of associated events upon synchronization

$$
a.P \mid a.Q \mid \bar{a}.R \;\rightarrow\; \begin{cases} a.P \mid Q \mid R \\ P \mid a.Q \mid R \end{cases}
$$

Pairings

Definition

A *pairing* is an association between occurrences of dual actions



Definition

A *determinization* of P along a pairing p is a renaming $\partial_p(P)$ of actions in P where names are equal only for related actions.

$$\partial_{p_1}(P) = a'.b'.\partial(A) \mid \bar{a}.c.\partial(B) \mid \bar{b}''.\bar{c}''.\partial(C) \mid a.\bar{c}$$

$$\partial_{p_2}(P) = a.b.\partial(A) \mid \bar{a}.c.\partial(B) \mid \bar{b}.\bar{c}.\partial(C) \mid a'.\bar{c}'$$

Pairings

Facts about pairings:

- each run induces a pairing
- runs are equivalent up to permutation of independent events iff they induce the same pairing
- if p is a consistent pairing of P then p is the unique maximal consistent pairing of $\partial_p(P)$

Hence pairings are *execution schedules* and determinized terms represent them inside the process language.

Logic will type these schedules.

A logic of schedules

Types of schedules:

$A, B :=$	$\langle a \rangle A$	do action a and then act as A
	$A \otimes B$	two independent parts, one as A, the other as B
	$A \parr B$	A and B are both exhibited, but correlated
	α	an unspecified behaviour
	$\alpha^\perp$	something that can interact with α

Transforming schedules:

$A_1, \ldots, A_n \vdash B$ behave as type B using one schedule of each type A_i

The role of the axiom rule

Two-sided presentation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{1 : \alpha \vdash \alpha}}{\bar{b} : \alpha \vdash \langle \bar{b} \rangle \alpha}}{a.\bar{b} : \alpha \vdash \langle a\bar{b} \rangle \alpha} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{1 : \alpha \vdash \alpha}}{\bar{c} : \alpha \vdash \langle \bar{c} \rangle \alpha}}{b.\bar{c} : \langle \bar{b} \rangle \alpha \vdash \langle \bar{c} \rangle \alpha}}{b.\bar{c} \vdash \langle \bar{b} \rangle \alpha \multimap \langle \bar{c} \rangle \alpha} \quad \dfrac{\dfrac{\dfrac{}{1 : \langle \bar{b} \rangle \alpha \vdash \langle \bar{b} \rangle \alpha} \quad \dfrac{\dfrac{\dfrac{}{1 : \alpha \vdash \alpha}}{d : \alpha \vdash \langle d \rangle \alpha}}{c.d : \langle \bar{c} \rangle \alpha \vdash \langle d \rangle \alpha}}{c.d : \langle \bar{b} \rangle \alpha, \langle \bar{b} \rangle \alpha \multimap \langle \bar{c} \rangle \alpha \vdash \langle d \rangle \alpha}}{\bar{a}.c.d : \langle a\bar{b} \rangle \alpha, \langle \bar{b} \rangle \alpha \multimap \langle \bar{c} \rangle \alpha \vdash \langle d \rangle \alpha}}{b.\bar{c} \mid \bar{a}.c.d : \langle a\bar{b} \rangle \alpha \vdash \langle d \rangle \alpha}}{a.\bar{b} \mid b.\bar{c} \mid \bar{a}.c.d : \alpha \vdash \langle d \rangle \alpha}$$

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$$\dfrac{b.\bar{c} \vdash \langle \bar{b} \rangle \alpha \multimap \langle \bar{c} \rangle \alpha \qquad \bar{a}.c.d : \langle a\bar{b} \rangle \alpha, \langle \bar{b} \rangle \alpha \multimap \langle \bar{c} \rangle \alpha \vdash \langle d \rangle \alpha}{b.\bar{c} \mid \bar{a}.c.d : \langle a\bar{b} \rangle \alpha \vdash \langle d \rangle \alpha}$$

$$\dfrac{a.\bar{b} : \alpha \vdash \langle a\bar{b} \rangle \alpha \qquad b.\bar{c} \mid \bar{a}.c.d : \langle a\bar{b} \rangle \alpha \vdash \langle d \rangle \alpha}{a.\bar{b} \mid b.\bar{c} \mid \bar{a}.c.d : \alpha \vdash \langle d \rangle \alpha}$$

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  \qquad
  \dfrac{
    \dfrac{\dfrac{\dfrac{\dfrac{}{1 \vdash \alpha^\perp, \alpha}}{\bar{c} \vdash \alpha^\perp, \langle \bar{c} \rangle \alpha}}{b.\bar{c} \vdash \langle b \rangle \alpha^\perp, \langle \bar{c} \rangle \alpha}}{b.\bar{c} \vdash \langle b \rangle \alpha^\perp \parr \langle \bar{c} \rangle \alpha}
    \qquad
    \dfrac{\dfrac{\dfrac{}{1 \vdash \langle b \rangle \alpha^\perp, \langle \bar{b} \rangle \alpha} \quad \dfrac{\dfrac{\dfrac{}{1 \vdash \alpha^\perp, \alpha}}{d \vdash \alpha^\perp, \langle d \rangle \alpha}}{c.d \vdash \langle c \rangle \alpha^\perp, \langle d \rangle \alpha}}{c.d \vdash \langle b \rangle \alpha^\perp, \langle \bar{b} \rangle \alpha \otimes \langle c \rangle \alpha^\perp, \langle d \rangle \alpha}}{\bar{a}.c.d \vdash \langle \bar{a}b \rangle \alpha^\perp, \langle \bar{b} \rangle \alpha \otimes \langle c \rangle \alpha^\perp, \langle d \rangle \alpha}
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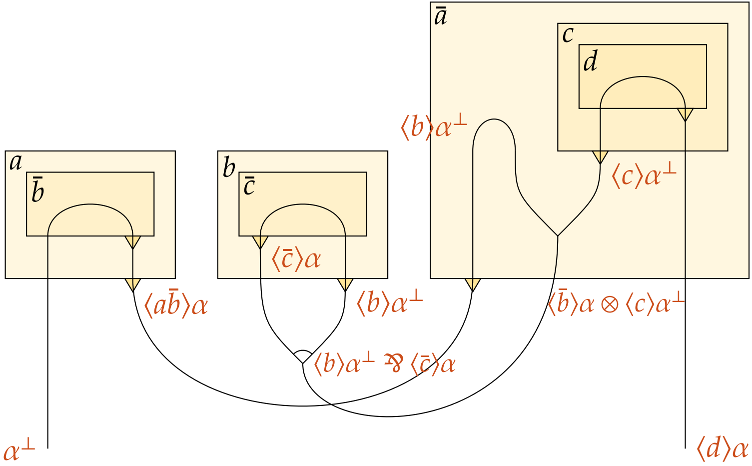
}{a.\bar{b} \mid b.\bar{c} \mid \bar{a}.c.d \vdash \alpha^\perp, \langle d \rangle \alpha}
$$

Duality: $(A \otimes B)^\perp = A^\perp \parr B^\perp, (\langle a \rangle A)^\perp = \langle \bar{a} \rangle (A^\perp)$.

The role of the axiom rule

Proof net presentation



$$a.\bar{b}.1 \mid (b.\bar{c}.1 \mid \bar{a}.c.d)$$

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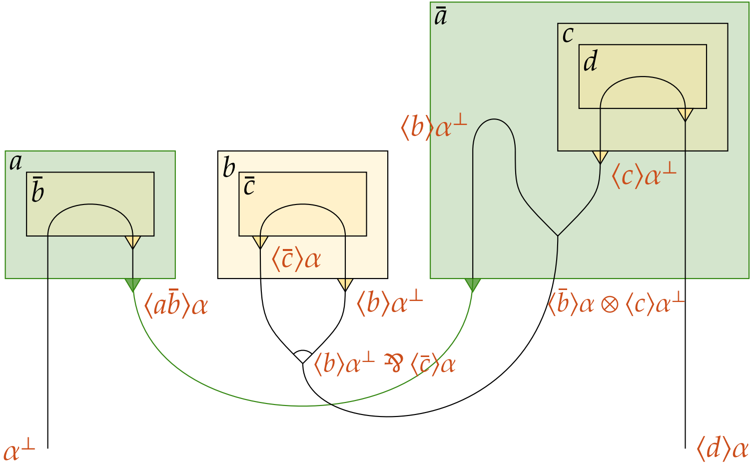

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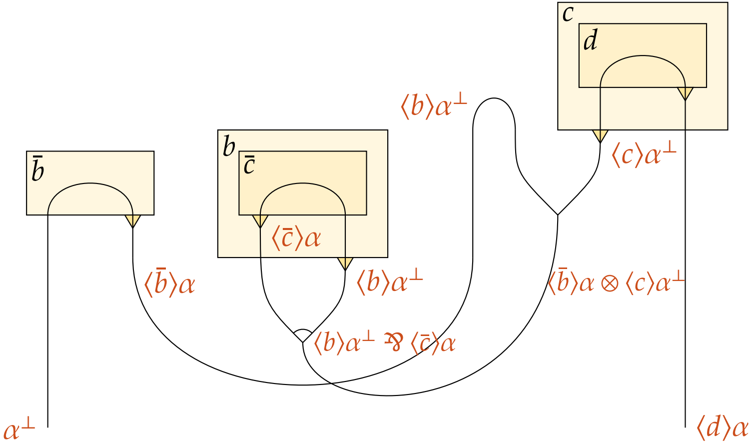

$$\bar{b}.1 \mid (b.\bar{c}.1 \mid c.d)$$

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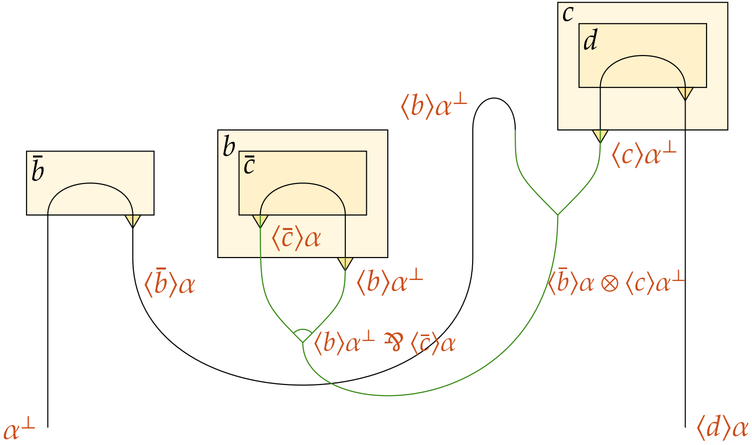

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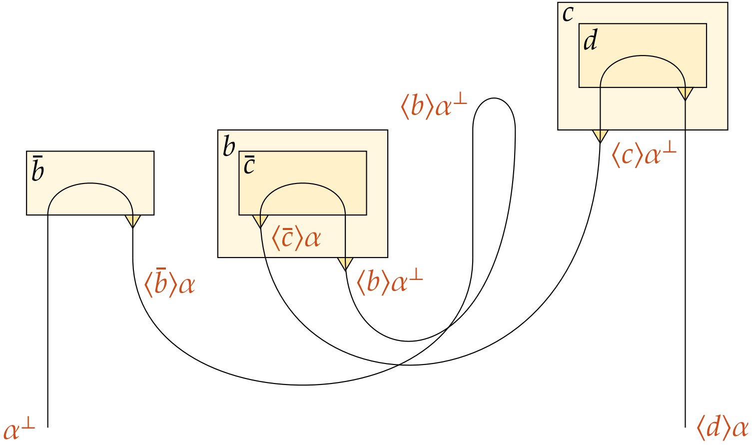

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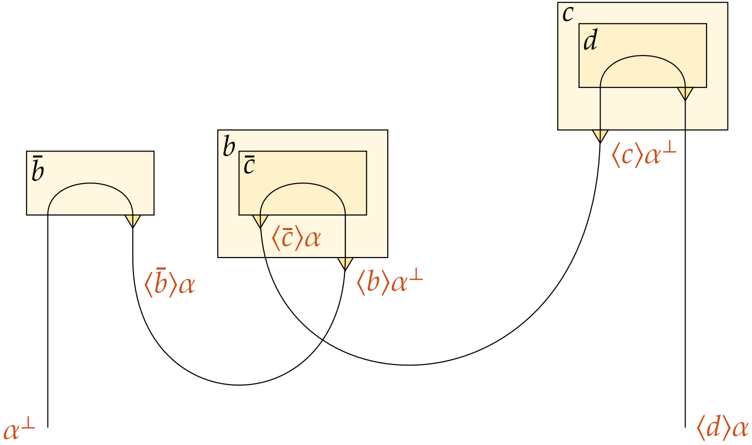

$$\bar{b}.1 \mid (b.\bar{c}.1 \mid c.d)$$

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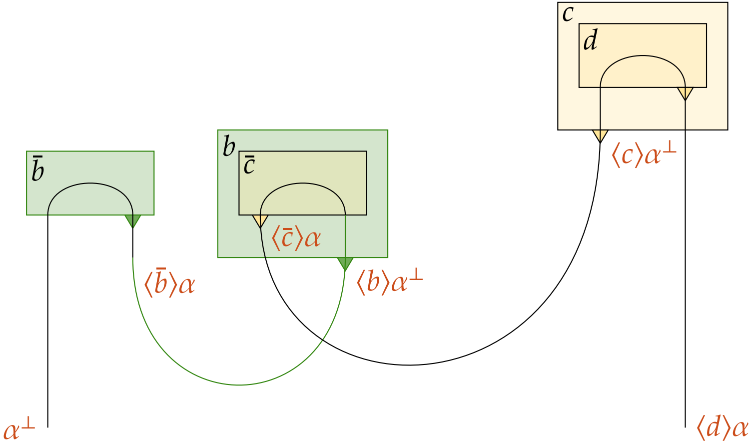
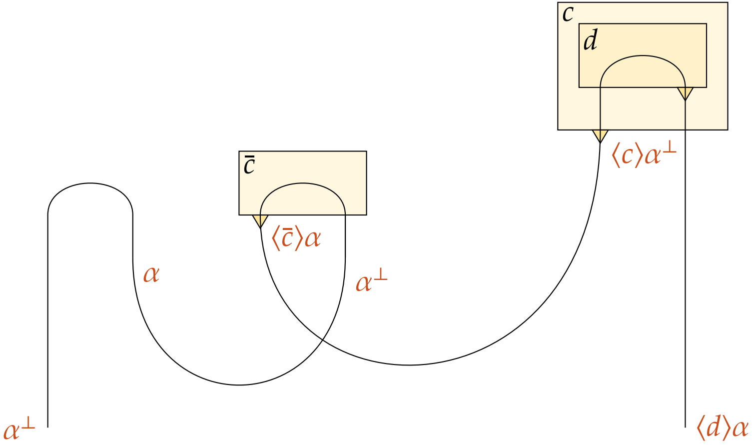


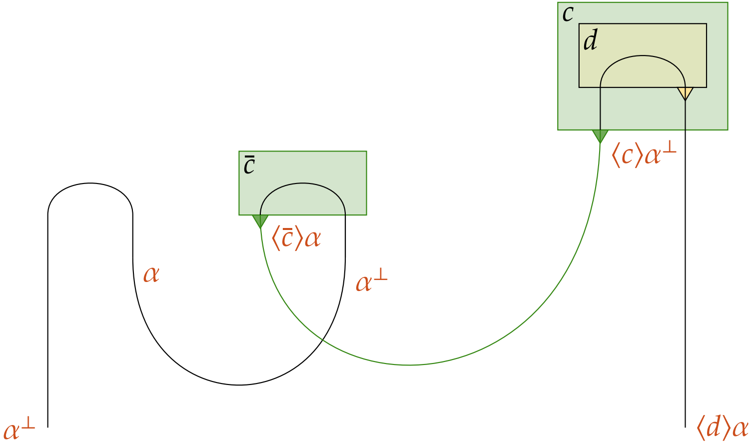
$\bar{b}.1 \mid (b.\bar{c}.1 \mid c.d)$



$$1 \mid (\bar{c}.1 \mid c.d)$$

The role of the axiom rule

Proof net presentation



$$1 \mid (\bar{c}.1 \mid c.d)$$

The role of the axiom rule

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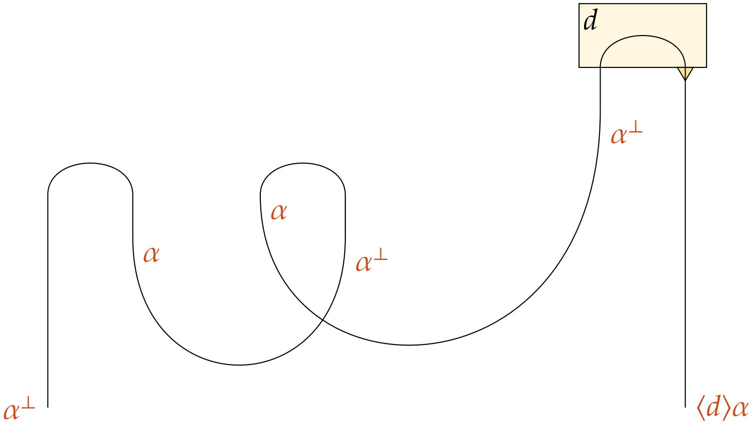


$$1 \mid (1 \mid d)$$

Mandatory theorems

Theorem (Soundness)

Typing is preserved by reduction,
head cut-elimination steps correspond to execution steps.

- a typed deterministic term cannot deadlock,
- normalization corresponds to a particular execution.

Mandatory theorems

Theorem (Soundness)

Typing is preserved by reduction,
head cut-elimination steps correspond to execution steps.

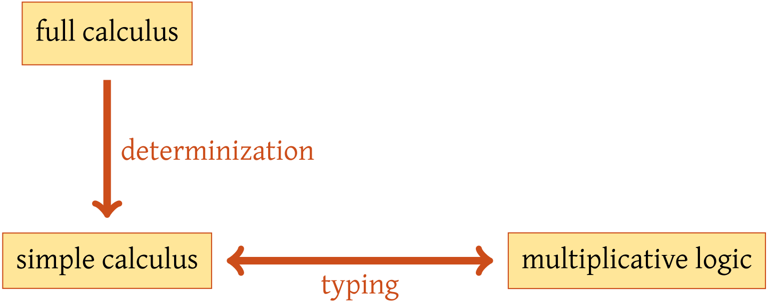
- a typed deterministic term cannot deadlock,
- normalization corresponds to a particular execution.

Theorem (Completeness)

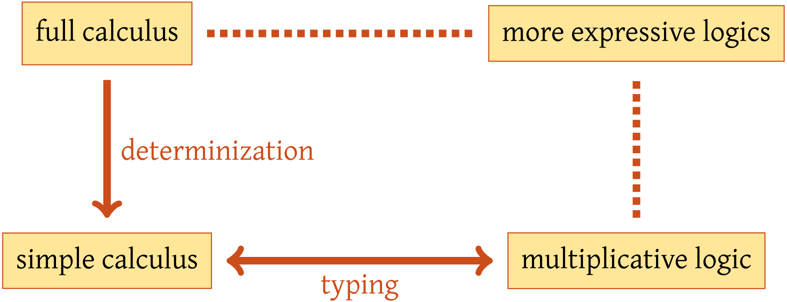
For every lock-avoiding run $P_1 \to \ldots \to P_n$ *there are corresponding typings such that* $\pi_1 : P_1 \vdash \Gamma \to \ldots \to \pi_n : P_n \vdash \Gamma$ *is a cut elimination sequence.*

- need to define “lock-avoiding”

Summing up


full calculus
determinization
simple calculus
typing
multiplicative logic

Summing up


full calculus
more expressive logics
determinization
simple calculus
typing
multiplicative logic

Conclusion, extensions

Current state of affairs:

- A logical description of scheduling in processes
  - describes how schedules can be safely composed
  - normal forms as basic *open* schedules
- Explicitation of *control flow* through processes
- Hints for a new study of *causality* in processes

Possible extensions:

- Connectives to combine related behaviours:

$$t_1.(t_2 + f_2 \mid \bar{t}_0) + f_1.(t_2.\bar{t}_0 + f_1.\bar{f}_0) \vdash B[t_1, f_1] \otimes B[t_2, f_2] \multimap B[t_0, f_0]$$
$$\text{where } B[t, f] := \alpha \multimap \langle \bar{t} \rangle \alpha \oplus \langle \bar{f} \rangle \alpha$$

- Predicates to describe states
- Richer action modalities for richer communication

Thank you.